MATH2060 TUT07

8. Let $F(x)$ be defined for $x \ge 0$ by $F(x) := (n-1)x - (n-1)n/2$ for $x \in [n-1, n)$, $n \in \mathbb{N}$. Show that F is continuous and evaluate $F'(x)$ at points where this derivative exists. Use this result to evaluate $\int_a^b [\![x]\!]dx$ for $0 \le a < b$, where $[\![x]\!]$ denotes the greatest integer in x, as defined in Exercise 5.1.4.

Ans: To show that f is continuous, it suffices to check that

$$\lim_{x \to n^-} F(x) = F(n) \qquad \forall n \in \mathbb{N}.$$

Indeed, $\lim_{x\to n^-} F(x) = \lim_{x \to n^-} \left[(n-1)x - (n-1)n/2 \right] = (n-1)n - (n-1)n/2 = \frac{(n-1)n}{2}$

$$F(n) = (n)n - n(n+1)/2 = \frac{n}{2}(2n-n-1) = \frac{(n-1)n}{2}$$

So f is cts on $[0, \infty)$.

$\forall n \in \mathbb{N}$,

$$\lim_{x\to n^-} \frac{F(x)-F(n)}{x-n} = \lim_{x \to n^-} \frac{(n-1)x - (n-1)n/2 - (n-1)n/2}{x-n}$$

$$= \lim_{x\to n^-} \frac{(n-1)(x-n)}{x-n} = n-1.$$

$$\lim_{x\to n^+} \frac{F(x)-F(n)}{x-n} = \lim_{x\to n^+} \frac{nx - n(n+1)/2 - (n-1)n/2}{x-n}$$

$$= \lim_{x\to n^+} \frac{n(x-n)}{x-n} = n$$

So F is not diff. at any $n \in \mathbb{N}$.

Hence $F'(x) = n-1$ for $x \in (n-1, n)$, $n \in \mathbb{N}$.

$\qquad\qquad = [\![x]\!]$

Now a) F is cts on $[a,b]$

b) $F'(x) = [\![x]\!] \quad \forall x \in [a,b] \setminus E$,

where $E := ([a,b] \cap \mathbb{N}) \cup \{0\}$ is a finite set

c) $[\![x]\!] \in \mathcal{R}[a,b]$ since it is a step fcn.

By FTC (1st form),

$$\int_a^b [\![x]\!]\, dx = F(b) - F(a)$$

$$= \left([\![b]\!] b - [\![b]\!]([\![b]\!]+1)/2 \right) - \left([\![a]\!] a - [\![a]\!]([\![a]\!]+1)/2 \right)$$

16. If $f : [0,\ 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0,\ 1]$, show that $f(x) = 0$ for all $x \in [0,\ 1]$.

Ans: Note $\forall x \in [0,1]$,

$$\int_0^x f = \int_x^1 f = \int_0^1 f - \int_0^x f$$

$$\Rightarrow \quad 2\int_0^x f = \int_0^1 f \qquad (*)$$

Since f is cts on $[0,1]$, it follows from FTC (2nd form) that

1) $F(x) := \int_0^x f$ is diff. on $[0,1]$, and

2) $F'(x) = f(x) \quad \forall x \in [0,1]$.

Hence, $(*)$ yields, $\forall x \in [0,1]$,

$$\frac{d}{dx}\left(2F(x)\right) = \frac{d}{dx}\int_0^1 f$$

$$2f(x) = 2F'(x) = 0$$

That is $f(x) = 0 \quad \forall x \in [0,1]$ //

3. Let f and g be bounded functions on $I := [a, b]$. If $f(x) \le g(x)$ for all $x \in I$, show that $L(f) \le L(g)$ and $U(f) \le U(g)$.

Ans: Let $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a,b]$.

Then, $\forall\, k = 1, \dots, n,$

$$f(x) \le g(x) \qquad \forall\, x \in [x_{k-1}, x_k].$$

$$\Rightarrow \quad \inf_{[x_{k-1}, x_k]} f \le \inf_{[x_{k-1}, x_k]} g$$

So $$\sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]} f\,(x_k - x_{k-1}) \le \sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]} g\,(x_k - x_{k-1})$$

i.e. $$L(f; P) \le L(g; P) \qquad (*)$$

Since $(*)$ is true for any $P \in \underline{\mathcal{P}([a,b])}$, ← set of all partitions of $[a,b]$.

and $L(f) := \sup\{L(f;P) : P \in \mathcal{P}([a,b])\}$

$L(g) := \sup\{L(g;P) : P \in \mathcal{P}([a,b])\}$ both exist

because f, g are bounded on $[a,b]$,

we have

$$L(f) \le L(g)$$

Similarly, we can show that

$$U(f) \le U(g)$$

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5. Let f, g, h be bounded functions on $I := [a, b]$ such that $f(x) \le g(x) \le h(x)$ for all $x \in I$. Show that if f and h are Darboux integrable and if $\int_a^b f = \int_a^b h$, then g is also Darboux integrable with $\int_a^b g = \int_a^b f$.

Ans: Since f and h are Darboux integrable, we have

$$L(f) = U(f) = \int_a^b f$$
$$L(h) = U(h) = \int_a^b h$$

By Ex 3, $f(x) \le g(x) \le h(x)$ $\forall x \in [a, b]$ implies that

$$L(f) \le L(g) \le L(h)$$
$$U(f) \le U(g) \le U(h)$$

Since $\int_a^b f = \int_a^b h$, we have

$$L(g) = U(g) = \int_a^b f = \int_a^b h.$$

Hence g is Darboux integrable with

$$\int_a^b g = \int_a^b f = \int_a^b h$$

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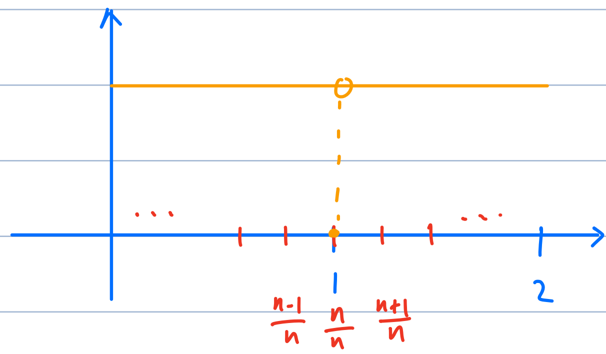
6. Let f be defined on $[0, 2]$ by $f(x) := 1$ if $x \neq 1$ and $f(1) := 0$. Show that the Darboux integral exists and find its value.

Ans: $\forall n \in \mathbb{N}$, let P_n be the partition of $[0,2]$ into $2n$ subintervals given by $P_n := \left(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{2n-1}{n}, \underbrace{\frac{2n}{n}}_{=2}\right)$.



Then, $$m_k = \inf_{[\frac{k-1}{n}, \frac{k}{n}]} f = 1$$
$$M_k = \sup_{[\frac{k-1}{n}, \frac{k}{n}]} f = 1 \qquad \forall k \in \{1, \ldots, 2n\} \setminus \{n, n+1\},$$

and $$\inf_{[\frac{n-1}{n}, \frac{n}{n}]} f = \inf_{[\frac{n}{n}, \frac{n+1}{n}]} f = 0, \quad m_n = m_{n+1} = 0$$
$$\sup_{[\frac{n-1}{n}, \frac{n}{n}]} f = \sup_{[\frac{n}{n}, \frac{n+1}{n}]} f = 1, \quad M_n = M_{n+1} = 1$$

Moreover, $x_k - x_{k-1} = \frac{1}{n}$.

So,
$$L(f; P_n) = (2n-2)(1)\frac{1}{n} + 2(0)\frac{1}{n} = 2 - \frac{2}{n}$$
$$U(f; P_n) = (2n-2)(1)\frac{1}{n} + 2(1)\frac{1}{n} = 2$$

Now, $L(f) = \sup\{L(f;P) : P \in \mathcal{P}([0,2])\} \geq L(f;P_n) = 2 - \frac{1}{n} \quad \forall n \in \mathbb{N}$

$\Longrightarrow \quad L(f) \geq 2$

$U(f) = \inf\{U(f;P) : P \in \mathcal{P}([0,2])\} \leq U(f;P_n) = 2$

Since $2 \leq L(f) \leq U(f) \leq 2$, we conclude that $L(f) = U(f) = 2$

So the Darboux integral of f exists and the value is 2 //